Birefringence by a smoothly inhomogeneous locally isotropic medium

V. S. Liberman and B. Ya. Zel'dovich
Nonlinear Optics Laboratory, Technical University, 454080 Chelyabinsk, Russia
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Propagation of an electromagnetic wave in a layered locally isotropic medium is considered. An effective birefringence $n_{\parallel}-n_{\perp}\sim(\lambda/a)^2$ for the $\parallel$ and $\perp$ eigenpolarizations appears, and the corresponding phase difference $\varphi_{\parallel}-\varphi_{\perp}\sim(\lambda/a)$ is calculated in this paper. An alternative method of calculation of the higher ($\sim\lambda/a$) WKB corrections for the one-dimensional Schrödinger problem is developed as a by-product of the work.

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I. INTRODUCTION

Propagation of light in a smoothly inhomogeneous medium is a subject of geometric optics [1–4]. We will limit ourselves by the case of locally isotropic medium, where the tensors $\varepsilon_{ik}(r)$ and $\mu_{ik}(r)$ of dielectric and magnetic susceptibilities are reduced to scalars:

$$\varepsilon_{ik}(\mathbf{r})=\varepsilon(\mathbf{r})\delta_{ik},\quad \mu_{ik}(\mathbf{r})=\mu(\mathbf{r})\delta_{ik}\ . \tag{1}$$

The most crude approximation is to neglect completely the polarization of a wave and to discuss the "corresponding" Helmholtz or Schrödinger-type equation

$$\Delta\Psi(\mathbf{r})+\frac{\omega^2}{c^2}n^2(\mathbf{r})\Psi=0\ , \tag{2}$$

where ω is the frequency of monochromatic wave $\psi(\mathbf{r},t)=\Psi(\mathbf{r})\exp(-i\omega t)$, and c is the speed of light in vacuum. We have also introduced in (2) the refractive index of our transparent medium:

$$n(\mathbf{r})=\sqrt{\varepsilon(\mathbf{r})\mu(\mathbf{r})}\ . \tag{3}$$

Asymptotic solution of a one-dimensional problem for Eq. (2) is given by the WKB method, or so-called quasiclassical approximation [5–7]. The details of WKB method for multidimensional problems are discussed in Refs. [8–10].

The most naive generalization of the Helmholtz equation (2) for the account of polarization degrees of freedom makes use of the local isotropy of the medium (1) and consists in a suggestion to make propagation and polarization independent. Such an approach has some applicability for the limiting case of an optical beam with very small ($\ll 1$ rad) angular deviations of the propagation direction from some fixed axis, i.e., for the paraxial approximation. However, that approach fails completely if the ray's angular deviation is more than or about 1 rad. In that case "conservation" of a polarization in "absolute" coordinate frame would contradict the transverse nature of electromagnetic waves.

The solution to that problem in the first nonvanishing approximation was given in the works by Bortolotti [11] in 1926, by Rytov [26] in 1938, by Vladimirsky [12] in 1941, by Berry [13] in 1984, and apparently in some others, see also [1,3,4], a recent review of Berry's phase [14], and experimental observations by Chiao and co-workers [15]. The equation for the ray in that approximation is the same as for a scalar wave:

$$\frac{d\mathbf{s}}{dl}=\nabla\ln n-\mathbf{s}(\mathbf{s}\cdot\nabla\ln n)\ . \tag{4}$$

Here $\mathbf{s}$ is approximately the unit tangent vector and l is the distance along the trajectory $\mathbf{r}=\mathbf{r}(l)$. The polarization unit vector $\mathbf{e}$ is assumed to be subjected to a parallel transport [11–13,26]. The idea of those works may be expressed in the following terms: let us try not to change the polarization vector $\mathbf{e}$, since the medium is locally isotropic, the only changes that we must introduce are those which keep the polarization transverse to the new propagation direction

$$\frac{d\mathbf{e}}{dl}=-\mathbf{s}\left[\mathbf{e}\cdot\frac{d\mathbf{s}}{dl}\right] \tag{5}$$

so that $(\mathbf{e}\cdot\mathbf{s})\equiv 0$ at any l. Thus the influence of the trajectory $\mathbf{s}(l)$ on the polarization $\mathbf{e}$ is governed by Eq. (5). For the sake of estimation of the order of magnitude it may be considered as resulting from a correction to the effective refractive index,

$$\frac{d\mathbf{e}}{dl}\sim\frac{\mathbf{e}}{a}\sim\frac{\omega}{c}\delta n_{\text{eff}}\mathbf{e},\quad \delta n_{\text{eff}}\sim\frac{c}{\omega a}=\frac{\lambda}{2\pi a}\ . \tag{6}$$

Here a is the spatial scale of inhomogeneity of the refractive index and of the ray (if $\delta n\sim n$).

A reverse effect of the influence of (circular) polarization on the trajectory was recently calculated [16–18] and observed experimentally [19,20], see also [21]. It was called "optical Magnus effect" or "optical Ping-Pong effect" and consists in a transverse shift of the ray in the process of refraction:

$$\frac{d\mathbf{r}}{dl}=\mathbf{s}+\frac{c}{\omega n}\sigma[\mathbf{s}\times\nabla\ln n]\equiv\mathbf{s}+\frac{c\sigma}{\omega n}\left[\mathbf{s}\times\frac{d\mathbf{s}}{dl}\right]\ , \tag{7}$$

where $\sigma=+1$ or -1 for right and left circular polarization, respectively. Thus both effects (5) and (7) appear as resulting from the corrections $\delta n_{\text{eff}}\sim(\lambda/a)$ for smooth variations of refractive index $n(\mathbf{r})$.

Two features of those corrections should be mentioned. The first feature is the conservation of the circularity $\sigma = i\mathbf{s}\cdot[\mathbf{e}\times\mathbf{e}^*]$ along the trajectory. In particular, the initial linear polarization ($\mathbf{e}=\mathbf{e}^*$) is kept linear during the propagation. That means that the Rytov-Vladimirsky-Berry-Chiao rotation of polarization may be considered as "geometric optical activity," i.e., as a circular birefringence, but without "usual" birefringence.

The second important feature is the purely geometric origin of those corrections. Namely, those effects (in the range of the validity of corresponding equations) are determined by the trajectory $\mathbf{r}(l)$ and hence by the refractive index profile $n(\mathbf{r})$ only, without being connected to $\varepsilon(\mathbf{r})$ or $\mu(\mathbf{r})$ separately. As a matter of fact, $\mu(\mathbf{r})=1$ with a very high precision in optics, but for radio waves both $\mu(\mathbf{r})$ and $\varepsilon(\mathbf{r})$ may be inhomogeneous.

To illustrate the point of the present paper, let us consider two optical devices with rather similar geometry of propagating rays. The first one is the Fresnel rhomb, Fig. 1(a). This device is usually adjusted to produce the $\pi/4$ phase shift between $\mathbf{e}_\parallel$ and $\mathbf{e}_\perp$ polarization in each (of two) step of total internal reflection. As a result the input linear polarization tilted $+45°$ to the ray's plane is transformed into the right circular output polarization, and the $-45°$ input polarization gives the output circle.

The second one is the piece of a planar multimode gradient-index waveguide cut off as a Fresnel rhomb, Fig. 1(b). Suppose that the waveguide is polarizationally neutral (made of a locally isotropic material) and really multimode, $a \gg \lambda$, where a is the core size and the ray's curvature radius scale. Then we know both from theory and from the experiment that the arbitrary input *linear* polarization is transferred into the *linear* output one, i.e., that such a gradient Fresnel rhomb possesses no birefringence.

The work of the Fresnel rhomb is connected with a very sharp change of dielectric susceptibility $\varepsilon(\mathbf{r})$ at the glass-surface–air interface. That sharpness means that $a \ll \lambda$ in the case of the Fresnel rhomb, so that the expansion into λ/a series is not valid anymore. Besides that, the hypothetical Fresnel rhomb made of a material with $\varepsilon(\mathbf{r})=1$, but $\mu(\mathbf{r})>1$, would give the opposite sign of output circularity, since the $+45°$ linear polarization of the *electric* vector of the incident wave corresponds just to $-45°$ linear polarization of the *magnetic* vector of that wave. Speaking very crudely, the aim of the present paper is to describe gradual transition from the device at Fig. 1(b) to that at Fig. 1(a). In other words, we want to describe the linear birefringence which appears as a correction $\delta n_\parallel - \delta n_\perp \sim (\lambda/a)^2$ or as a phase shift $\varphi_\parallel - \varphi_\perp \sim (\lambda/a)$. The above-mentioned example of a hypothetical "magnetic" Fresnel rhomb leads us to the conclusion that here we must also take into account the spatial variation of the so-called wave resistance

$$\rho(\mathbf{r}) = \sqrt{\mu(r)/\varepsilon(r)} \tag{8}$$

along with the spatial profile of refractive index $n(\mathbf{r})$ from (2).

Another example shows that, unlike the circular birefringence $\sim(\lambda/a)$ in (5) and (7), linear birefringence $\sim(\lambda/a)^2$ is not a geometrical effect caused by the ray's behavior only. Let us remember that Chiao and coworkers used a curved monomode fiber instead of curved ray to measure Berry's phase [15] (or the geometrical rotation of the polarization). If one tried to measure linear birefringence in a smoothly bent fiber instead of our gradient Fresnel rhomb the result would be quite different [22].

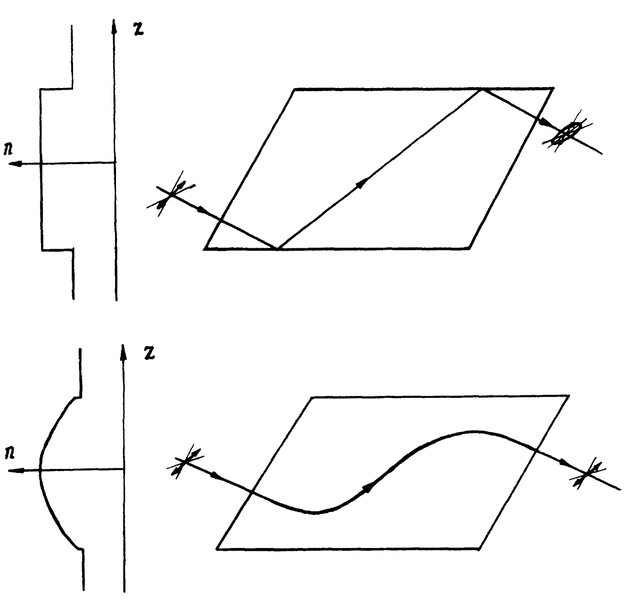

FIG. 1. (a) Birefringence in a Fresnel rhomb. (b) A gradient analog of the Fresnel rhomb with very small ($\sim\lambda/a$) birefringence.

II. BASIC EQUATIONS

Taking monochromatic time dependence as $\exp(-i\omega t)$, we reduce Maxwell equations for the complex vector amplitudes of electric ($\mathbf{E}$) and magnetic ($\mathbf{H}$) fields to the system

$$\mathrm{rot}\mathbf{E} = i\frac{\omega}{c}\mu\mathbf{H}, \quad \mathrm{rot}\mathbf{H} = -i\frac{\omega}{c}\varepsilon\mathbf{E} . \tag{9}$$

The well-known conditions $\mathrm{div}(\varepsilon\mathbf{E}) = \mathrm{div}(\mu\mathbf{H}) = 0$ are not independent equations, but follow from (9). We will consider the case of a layered medium, i.e., the one where all the parameters $[\varepsilon, \mu, n=(\varepsilon\mu)^{1/2}, \rho=(\mu/\varepsilon)^{1/2}]$ depend on one coordinate only, and it will be denoted by z. For such a medium the solution of the system (9) may be taken in the form $\exp(ik_x x + ik_y y)f(z)$. By a proper choice of orientation in the (x,y) plane the value of k_y may be reduced to zero, and it will be assumed so. Then the value of k_x is convenient to write as $k_x = (\omega/c)\beta$, where $\beta = \sin\alpha_{\mathrm{air}}$ may be interpreted as the sinus of the incidence angle in the air. Thus we may look for a solution

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}(z)\exp\left[i\frac{\omega}{c}\beta x\right], \quad \mathbf{H}(\mathbf{r}) = \mathbf{H}(z)\exp\left[i\frac{\omega}{c}\beta x\right] \tag{10}$$

and for the amplitudes $\mathbf{E}(z)$, $\mathbf{H}(z)$ we get from (9)

$$-\frac{dE_y}{dz}=i\frac{\omega}{c}\mu H_x, \quad -\frac{dH_y}{dz}=-i\frac{\omega}{c}\varepsilon E_x , \tag{11}$$

$$\frac{dE_x}{dz}-i\frac{\omega}{c}\beta E_z=i\frac{\omega}{c}\mu H_y, \quad \frac{dH_x}{dz}-i\frac{\omega}{c}\beta H_z=-i\frac{\omega}{c}\varepsilon E_y , \tag{12}$$

$$\beta E_y=\mu H_z, \quad \beta H_y=-\varepsilon E_z . \tag{13}$$

It is convenient to exclude H_z and E_z and to get two pairs of first-order coupled equations. One of them governs E_y polarization:

$$\frac{dE_y}{dz}=-i\frac{\omega}{c}\mu H_x, \quad \frac{dH_x}{dz}=-i\frac{\omega}{c}\frac{1}{\mu}(n^2-\beta^2)E_y \tag{14}$$

and the other deals with H_y polarization,

$$\frac{dH_y}{dz}=i\frac{\omega}{c}\varepsilon E_x, \quad \frac{dE_x}{dz}=i\frac{\omega}{c}\frac{1}{\varepsilon}(n^2-\beta^2)H_y . \tag{15}$$

Sections III–V and VII of the present paper are devoted to the approximate solutions of (14) and (15) for the case $\lambda \ll a$, where $a \equiv \delta z$ is the characteristic size of the smooth inhomogeneity of the parameters $n(z)$ and $\rho(z)$.

III. NO TURNING POINT CASE: SLOWLY VARYING ENVELOPE APPROXIMATION

Suppose that the refractive index $n(z)$ and the incidence angle $\alpha_{\text{air}}=\arcsin\beta$ are chosen in such a way that the function $n^2(z)-\beta^2$ is always positive and does not take very small values. Besides, we assume that $n(z)\rightarrow n_1$ at $z\rightarrow-\infty$ and $n(z)\rightarrow n_2$ at $z\rightarrow+\infty$. Then the trajectory of a ray propagating from $z=-\infty$ to $z=+\infty$ looks like the one shown in Fig. 2, and there is no turning point.

Application of classical Snell law would give for the local "refraction angle" $\alpha(z)$

$$\sin\alpha(z)=\frac{\beta}{n(z)} , \qquad \cos\alpha(z)=\gamma(z)=\left[1-\frac{\beta^2}{n^2(z)}\right]^{1/2} . \tag{16}$$

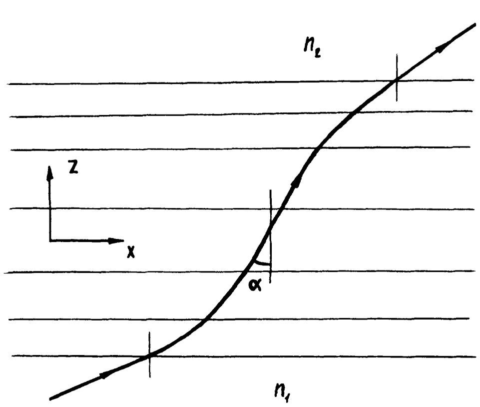

FIG. 2. Propagation in a layered medium without turning point.

We will use (16) as a definition to be used in the new form of amplitude representation. In the same classical approach the Poynting vector $\mathbf{S}=([\mathbf{E}\times\mathbf{H}^*]+[\mathbf{E}^*\times\mathbf{H}])c/16\pi$ must have constant z component. Since $\varepsilon|\mathbf{E}|^2\approx\mu|\mathbf{H}|^2$ in monodirectional wave, one gets $S_z=(c/8\pi)|\mathbf{E}|^2(\varepsilon/\mu)^{1/2}\cos\alpha=\text{const}$. Besides that, with the same precision $|E_x|=|\mathbf{E}|\cos\alpha(z)$ for E_x polarization and $|E_y|=|\mathbf{E}|$ for E_y polarization.

In the same approximation, z component of the local wave vector $k_x=(\omega/c)n(z)\cos\alpha(z)=(\omega/c)\sqrt{n^2(z)-\beta^2}$, and z dependence of the phase is given by $\exp[i(\omega/c)\int^z dz'\sqrt{n^2(z')-\beta^2}]$.

All that suggests the idea to introduce the couple of new unknown variables $A_y(z)(z)$ and $A_x(z)(z)$ as the slowly varying amplitudes:

$$E_y(z)=A_y(z)\sqrt{\rho(z)/\gamma(z)} \times\exp\left[i\frac{\omega}{c}\int^z\sqrt{n^2(z')-\beta^2}dz'\right] , \tag{17}$$

$$E_x(z)=A_x(z)\sqrt{\rho(z)\gamma(z)} \times\exp\left[i\frac{\omega}{c}\int^z\sqrt{n^2(z')-\beta^2}dz'\right] , \tag{18}$$

for which we can get the second-order equations. Direct substitution gives an equation which originally contains 28 terms. However, after long work on cancellation and reduction of similar terms we were able to get

$$\frac{d^2A_y}{dz^2}+\frac{dA_y}{dz}2i\frac{\omega}{c}n\gamma\left[1+\frac{ic}{2\omega n\gamma^3}\frac{d\ln n}{dz}\right] + A_y(z)D_y(z)=0 , \tag{19}$$

$$\frac{d^2A_x}{dz^2}+\frac{dA_x}{dz}2i\frac{\omega}{c}n\gamma\left[1+\frac{5ic}{2\omega n\gamma^3}\frac{d\gamma}{dz}+\frac{ic}{\omega n\gamma^2}\frac{d\ln n}{dz}\right] + A_x(z)D_x(z)=0 , \tag{20}$$

$$D_y(z)=\left[\left[\frac{d\ln n}{dz}\right]^2\frac{5-4\gamma^2-\gamma^4}{4\gamma^4}-\frac{1}{4}\left[\frac{d\ln\rho}{dz}\right]^2 -\frac{1}{2}\frac{d\ln n}{dz}\frac{d\ln\rho}{dz}+\frac{1}{2}\frac{d^2\ln\rho}{dz^2}\right] , \tag{21}$$

$$D_x(z)=\left[\left[\frac{d\ln n}{dz}\right]^2\frac{-7+8\gamma^2+9\gamma^4}{4\gamma^4}-\frac{1}{4}\left[\frac{d\ln\rho}{dz}\right]^2 +\frac{\gamma^2-2}{\gamma^2}\frac{d\ln n}{dz}\frac{d\ln\rho}{dz}+\frac{1}{2}\frac{1-\gamma^2}{\gamma^2}\frac{d^2\ln n}{dz^2}\right] . \tag{22}$$

We see that the order of magnitude of D_y and D_x is about a^{-2}, where a is the spatial scale of inhomogeneity. The assumption of slow dependence of the amplitudes $A_x(z)$ and $A_y(z)$ on coordinate $z[dA/dz\sim(\lambda/a^2)A]$ allows us to neglect d^2A/dz^2 in (19), (20) and to take the terms $\propto dA_x/dz$, dA_y/dz only with the main part $2i\omega n\gamma/c$ of the coefficient. As a result we get the first-order equations

$$2i\frac{\omega}{c}n\gamma\frac{dA_y}{dz}=D_y A_y,\quad 2i\frac{\omega}{c}n\gamma\frac{dA_x}{dz}=D_x A_x \ . \tag{23}$$

These equations have evident solutions; for example, one gets for $A_y(z)$

$$A_y(z)=A_y(-\infty)\exp\left[-i\frac{c}{2\omega}\int_{-\infty}^{z}\frac{D_y(z')}{n(z')\gamma(z')}dz'\right] . \tag{24}$$

We are interested in this paper in finding the effective birefringence, i.e., the phase difference between the x- and y-polarized waves. Some of the contributions to φ_x and φ_y are canceled, and finally

$$\frac{d}{dz}[\varphi_y(z)-\varphi_x(z)]=\frac{c\beta^2}{2\omega n^3\gamma^3}\frac{d\ln n}{dz}\frac{d\ln\rho}{dz}+\frac{d}{dz}\psi(\beta,z) , \tag{25}$$

where

$$\psi=-\frac{c}{2\omega n}\left[\frac{1-\gamma^2}{\gamma^3}\frac{d\ln n}{dz}-\frac{1}{4\gamma}\frac{d\ln\rho}{dz}\right] . \tag{26}$$

Equation (25) constitutes the main result of the present section. It is worth noting that the phase difference accumulated at the interval from $z\to-\infty$ (where $n=n_1=\text{const}$, $\rho=\rho_1=\text{const}$), to $z\to+\infty$ (where $n=n_2=\text{const}$, $\rho=\rho_2=\text{const}$) is not influenced by the term $d\psi/dz$ with the full derivative.

It is evident that $\varphi_x-\varphi_y\equiv 0$, if $\beta=0$ (normal incidence), since in that case we have complete symmetry of rotation around the z axis, i.e., x and y directions are equivalent. Equation (25) shows also that the accumulated part of birefringence vanishes, if the wave resistance is homogeneous, $\rho=\text{const}$. It was more or less evident beforehand, since the accumulated shift $\varphi_x=\varphi_y$ between E_x and E_y is essentially the phase difference between H_y and E_y; however, for $\rho=\text{const}$ there is complete symmetry relative to a substitution $\mathbf{E}\to\mathbf{H},\mathbf{H}\to-\mathbf{E}$. What is less evident is that in the order $\delta\varphi\sim(\lambda/a)$ the accumulated phase difference $\varphi_x-\varphi_y$ is zero for a medium with variable $\rho(z)$, but with $n=\text{const}$. A very important case $\mu\equiv 1$, $\varepsilon=\varepsilon(z)$, $\rho(z)=n^{-1}(z)$ deserves special discussion. In that case $d\ln\rho/dz=-d\ln n/dz$, and the accumulated phase difference $\varphi(E_y)-\varphi(E_x)$ turns out to be negative.

IV. GENERAL CASE WITH A TURNING POINT

The total internal reflection (TIR) phenomenon corresponds to the trajectory of a ray shown in Fig. 3. In that case there is some particular value z_0, at which TIR occurs according to the classical geometric optics. It is determined by the equation

$$n^2(z_0)-\beta^2=0 . \tag{27}$$

Calculation of the phase difference should include doubling of the integral of Eq. (25) since the same interval at z axis is passed twice, see Fig. 3. The trajectory of a ray has parabolic form in the vicinity of the TIR point:

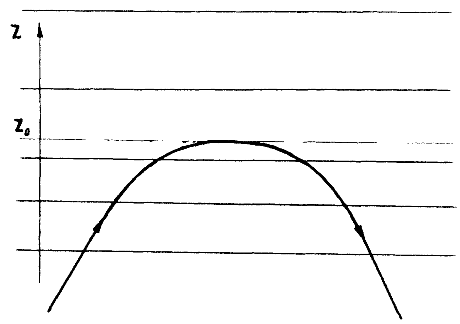

FIG. 3. Propagation in a layered medium with turning point.

$$z(x)=z_0-(z-x_0)^2\frac{g}{2n(z_0)} , \tag{28}$$

$$g=-\left.\frac{dn}{dz}\right|_{z_0}>0 . \tag{29}$$

Unfortunately, near that point the integral of $2d(\varphi_y-\varphi_x)/dz$ becomes divergent:

$$\varphi_y-\varphi_x\approx-\frac{2c\beta^2 g}{\omega n^4(2ng)^{3/2}}\left.\frac{d\ln\rho}{dz}\right|_{z_0}\int_{z_1}^{z_0}\frac{dz}{(z_0-z)^{3/2}}=\infty . \tag{30}$$

We would like to make an even stronger statement. If one would try to calculate φ_x and φ_y separately from the slowly varying amplitude first-order equations (23), the corresponding integrals would diverge even stronger—as $\int(z_0-z)^{-5/2}dz$. It means that one cannot neglect second-order derivatives in (19) and (20) in the vicinity of the TIR point. This circumstance is well known in optics and in quantum mechanics. Interference of "incident" and "totally reflected" waves produces a characteristic picture which is depicted by the Airy function, known from many papers and textbooks on quantum mechanics.

We will not consider the very special case when either $\varepsilon(z^*)=0$ or $\mu(z^*)=0$ at some point z^*. The singularity which occurs in such a case is discussed in detail in Ref. [3]. Then the only place which is singular from the point of view of geometric optics is z_0, the root of Eq. (27). In the vicinity of that point one can use approximation

$$n^2(z)-\beta^2\approx-2(z-z_0)gn(z_0) \tag{31}$$

and reduce the wave equation to the standard Airy equation. The small difference which arises in the process of such a reduction is just the aim of the next section.

V. REDUCTION TO THE SCHRÖDINGER-HELMHOLTZ SCALAR EQUATIONS

The coupled first-order equations (14) may be transformed into a single second-order equation

$$\mu\frac{d}{dz}\left[\frac{1}{\mu}\frac{dE_y}{dz}\right]+\frac{\omega^2}{c^2}[n^2(z)-\beta^2]E_y=0 . \tag{32}$$

In an analogous way the pair (15) may be transformed into

$$\varepsilon \frac{d}{dz}\left[\frac{1}{\varepsilon}\frac{dH_y}{dz}\right]+\frac{\omega^2}{c^2}[n^2(z)-\beta^2]H_y=0 . \tag{33}$$

It is convenient to make a transformation of independent variable

$$z_E=\int^z \mu(z')dz' \tag{34}$$

for which Eq. (32) takes the form of a one-dimensional Schrödinger or Helmholtz equation:

$$\frac{d^2E_y}{dz_E^2}+k_E^2(z_E)E_y=0 , \quad k_E^2(z_E)=\frac{\omega^2}{c^2\mu^2(z)}[n^2(z)-\beta^2] . \tag{35}$$

Similar substitution

$$z_H=\int^z \varepsilon(z')dz' \tag{36}$$

yields

$$\frac{d^2H_y}{dz_H^2}+k_H^2(z_H)H_y=0 , \quad k_H^2(z_H)=\frac{\omega^2}{c^2\varepsilon^2(z)}[n^2(z)-\beta^2] . \tag{37}$$

While the particular behavior of functions $z_B(z)$ and $z_H(z)$ is different, the only turning point in both cases corresponds to $z=z_0$ from (27).

Thus we must find the phase of the total reflection coefficient from a smooth barrier in the Schrödinger-type equation (36) or (37) with the effective potential energy $U(z)=[k_0^2-k^2(z)]\hbar^2/2m$ so that $U(z)\to 0$ at $z\to-\infty$ and the kinetic energy of the incident particle $\hbar^2k_0^2/2m$; here $k_0=k(z\to-\infty)$. This problem has exact solution only for a very limited number of types of potentials $U(z)$, see, e.g., Ref. [5–7]. Thus we need some approximate methods for the calculation of that phase with the accuracy of the order $(\lambda/a)^1$.

The well-known WKB approximation gives the next expression for that phase:

$$\varphi=2\int_{-\infty}^{z_0}[k(z)-k_0]dz-\pi/2 , \tag{38}$$

where the integral-type term is the ordinary quasiclassical (geometric-optical) phase, and $-\pi/2$ is the Airy-type correction due to a turning point. However, that expression (38) has the precision $\sim(\lambda/a)^0$ rad, i.e., the error of Eq. (38) is about $(\lambda/2\pi a)=(k_0a)^{-1}$, just about the value we need.

In the process of doing this work we were glad to make the reduction to Eqs. (35) and (37). We assumed that as the next step we would take the expression of our phase with the necessary, $\sim(\lambda/a)^1$, precision from the quantum mechanics. Unfortunately all the papers on the higher WKB expansions we have found [8–10,23,24] give this phase as some complex contour integral or, at best, reduce it to the derivative over the classical energy of the integral along the real axis [23]. Such a presentation of the result was not convenient for our needs. That encouraged us to derive it again through another technique. The corresponding results, formally equivalent to the ones from Refs. [23,24], are presented in the next section.

VI. CORRECTIONS $\sim(\lambda/a)$ TO THE WKB PHASE OF REFLECTION

The results of this section were obtained in collaboration with A. Yu. Savchenko and will be presented in an enlarged version separately. The calculation of the reflection phase for some given profile $k^2(z)$ is based on the idea of using as the basis the exact solution of some sample equation. One (but not the single one) example of such a sample problem is the quantum-mechanical motion in a homogeneous field, $U(x)=Fx$, which corresponds to the well-known Airy equation

$$\frac{d^2\psi}{dx^2}-x\psi(x)=0 \tag{39}$$

after the proper adjustment of the scale.

We will discuss here a slightly more general method of consideration of an arbitrary "sample" profile $k_x^2(x)$ to model the solution of the equation with the "physical" profile $k_z^2(z)$:

$$\frac{d^2\Psi(x)}{dx^2}+k_x^2(x)\Psi(x)=0 , \tag{40}$$

$$\frac{d^2\psi(z)}{dz^2}+k_z^2(z)\psi(z)=0 . \tag{41}$$

For definiteness we will choose the origins of x and z in such a way that $x<0$ and $z<0$ are the classically allowed regions, $x>0$ and $z>0$ are classically forbidden areas, and $x=0$ and $z=0$ are the corresponding turning points, Fig. 4.

Let us consider now two linearly independent real-valued solutions $\Psi_1(x)$ and $\Psi_2(x)$ of the "sample" equation (40). We will choose $\Psi_1(x)$ in such a way that it is a solution of the reflection problem: $\Psi_1(x\to+\infty)\to 0$. Then its asymptotic behavior for $x\to-\infty$ is [5–7]

$$\Psi_1(x)\approx[k_x(x)]^{-1/2}\cos[\zeta(x)] , \quad \zeta=\int_x^0 k_x(x')dx'-\frac{\pi}{4}-\delta_x . \tag{42}$$

The normalization in (42) is chosen for convenience of subsequent calculations. Small value δ_x is the correction $\sim(\lambda/a)$ to the WKB expression for the phase of reflection; it is assumed to be known for the "sample" problem with $k_x^2(x)$ profile. As the second solution we will take the one with the asymptotic behavior for $x\to-\infty$,

$$\Psi_2(x)\approx[k_x(x)]^{-1/2}\sin[\zeta(x)] . \tag{43}$$

For such solutions the Wronsky determinant $\Psi_1'(x)\Psi_2(x)-\Psi_1(x)\Psi_2'(x)$ is exactly equal to 1. It follows therefrom that $\Psi_2(x)$ grows exponentially for $x\to+\infty$.

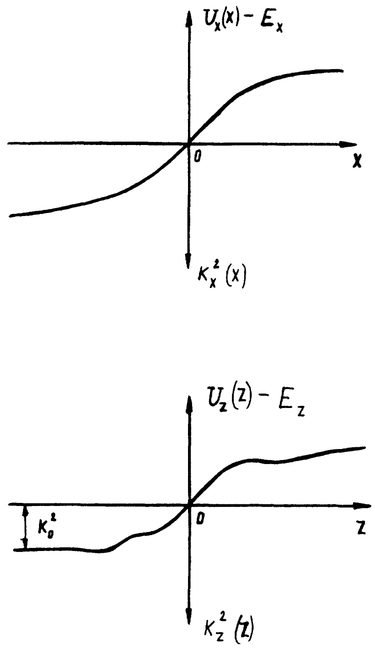

FIG. 4. "Sample" potential (a) and "physical" potential (b) for the calculation of the WKB phase of reflection.

The solution of the reflection problem for the "physical" profile $k_z^2(z)$ can be presented in the form

$$\psi(z)=C_1(z)f(z)\Psi_1(x(z))+C_2(z)f(z)\Psi_2(x(z)) \ . \tag{44}$$

Here the transformation of variables $x=x(z)$ is defined by the equalization of the classical action values (i.e., quasiclassical phases) accumulated from the turning points:

$$\int_{x(z)}^{0} k_x(x')dx'=\int_{z}^{0}k_z(z')dz' \ , \tag{45}$$

$$f(z)=\left[\frac{dx}{dz}\right]^{1/2}\equiv\left[\frac{k_x(x)}{k_z(z)}\right]^{1/2} . \tag{46}$$

We use the factor $f(z)$ so that the wave functions (44) with $C_1=\text{const}$, $C_2=\text{const}$ are just the quasiclassical (WKB) solutions [5–7] to the physical problem (41) with correct preexponential. Despite $k_x(x)$ and $k_z(z)$ being singular in the vicinity of the turning points $x=0$ and $z=0$ (as $\sqrt{-x}$ and $\sqrt{-z}$, respectively), the transformations $x=x(z)$ or $z=z(x)$ according to (45) are smooth analytical and mutually univalued near that point. In particular, $dx/dz|_0=[F_z(0)/F_x(0)]^{1/3}$, where $F_x=-d(k_z^2)/dz$, $F_x=-d(k_x^2)/dx$. Such a transformation of independent variable (45) and representation (44) of the wave function was repeatedly discussed in mathematical and physical literature [10,27–29].

Even $C_1(z)\equiv 1$, $C_2(z)\equiv 0$ gives quite a good approximation for the wave function. We are going to find small corrections to it. To solve this problem one can use the method of variation of "arbitrary constants" $C_1(z)$ and $C_2(z)$ with some additional relation between dC_1/dz and dC_2/dz. The relation

$$\Psi_1(x(z))\frac{dC_1}{dz}+\Psi_2(x(z))\frac{dC_2}{dz}=0 \tag{47}$$

proved to be most convenient. [It is worth noting that assuming another relation, $\Psi_1 d(fC_1)/dz+\Psi_2 d(fC_2)/dz=0$ instead of (47), we will also get the set of the first-order linear differential equations for $C_1(z)$ and $C_2(z)$, but their approximate solution with the same accuracy is much more difficult.] As a result we transform Eq. (41) into the coupled first-order pair

$$\frac{dC_1}{dz}+\Gamma(z)C_1=L(z)C_2, \quad \frac{dC_2}{dz}-\Gamma(z)C_2=M(z)C_1 \ , \tag{48}$$

where

$$\Gamma(z)=f\frac{d^2f}{dz^2}\Psi_1\Psi_2, \quad L=-f\frac{d^2f}{dz^2}\Psi_2^2, \quad M=f\frac{d^2f}{dz^2}\Psi_1^2 \ . \tag{49}$$

Let us assume that $F_x/F_z\sim 1$. Then $\lambda_x\sim\lambda_z$ and the order of magnitude of the coefficients Γ, L, M (of the inverse length dimensions) is λ/a^2. That allows us to drop all terms $\sim\Gamma$ and to take $C_1\equiv 1$ in the first order of perturbation theory and to get the first nonvanishing correction to the wave function in the form

$$C_2(z)=C_2(+\infty)-\int_z^{+\infty}\Psi_1^2(x(z'))f\frac{d^2f}{dz'^2}dz' \ . \tag{50}$$

The condition of the finiteness of the wave function (44) at $z\to+\infty$ (deeply inside the barrier) implies that $C_2(+\infty)=0$. Then outside the barrier, i.e., for $z\to-\infty$, the wave function (44) of our "physical" problem gets the asymptotic form

$$\psi(z)=\frac{1}{\sqrt{k^2(z)}}\cos\left[\int_z^0 k_z(z')dz'-\frac{\pi}{4}-\delta_z\right] ,$$

$$\delta_s=\delta_x+C_2(-\infty) \ . \tag{51}$$

Thus $C_2(-\infty)$ is the desired correction to the phase of reflection for the "physical" profile $k_z^2(z)$ in comparison with that for the "sample" profile $k_x^2(x)$. The result (50), (44), may be further simplified without loss of accuracy by making the substitution $\Psi_1^2(x)\to\langle\Psi_1^2(x)\rangle\approx[2k_x(x)]^{-1}$ for the allowed region $x<0$ and by taking $\Psi_1^2(x)\approx 0$ in the classically forbidden region $x>0$. Then we get

$$\delta_z=\delta_x-\frac{1}{2}\int_{-\infty}^{0}(k_xf)^{-1}\frac{d^2f}{dz^2}dz \ . \tag{52}$$

This is the main result of the present section, which was obtained by the authors in collaboration with A. Yu. Savchenko.

Important particular variant of Eq. (52) may be obtained for the "sample" profile $k_x^2(x)=-Fx$ with $F=\text{const}$. As is well known [5–7], $\delta_x\equiv 0$ for that case, since the extra phase shift exactly equals $-\pi/4$ for the solution of the Airy equation. The transformation of variables (45) acquires the form

$$-x(z)=\left[\frac{3}{2\sqrt{F}}\int_z^0 k_z(z')dz'\right]^{2/3} . \quad (53)$$

However, the simplification of Eq. (52) consists in the assumption $\delta_x=0$ only.

VII. BIREFRINGENCE IN THE CASE OF A TURNING POINT

Now we are equipped for the calculation of $\varphi(E_y)-\varphi(H_y)$ for the case of trajectory with the total internal reflection point, Fig. 3. The main parts of the phases are identically equal,

$$\int_{z_B(z)}^0 k_E(z_E')dz_E'-\frac{\pi}{4}=\int_{z_H(z)}^0 k_H(z_H')-\frac{\pi}{4}$$
$$=\frac{\omega}{c}\int_z^0[n^2(z')-\beta^2]^{1/2}dz'-\frac{\pi}{4} . \quad (54)$$

However, the small corrections $\delta_z(E_y)$ and $\delta_z(H_y)$ which must be calculated via (53) and (52) with $\delta_x=0$, turn out to be different. The corresponding result may be achieved after the calculations, which are simple in principle but are quite tedious. Here it is:

$$\varphi(E_y)-\varphi(E_x)=2\delta_z(E_y)-2\delta_z(E_x)$$
$$=-\int_{-\infty}^0\frac{c}{\omega}\frac{1}{\sqrt{n^2(z)-\beta^2}}\times\left[\frac{d\ln n}{dz}\frac{d\ln\rho}{dz}-\frac{d^2\ln\rho}{dz^2}\right]dz . \quad (55)$$

We want to emphasize that Eq. (55) gives the convergent integral in the vicinity of the turning point $z=0$.

At first sight, the second term proportional to $d^2\ln\rho/dz^2$ in the right-hand side of Eq. (55) contradicts the conclusion of Sec. IV where we have declared the absence of a accumulated birefringence in a medium with $n=\text{const}$, $\rho=\rho(z)$. However, for $n=\text{const}$ there is no turning point at all. Integration of that term over the interval between the regions with $\rho=\rho_1=\text{const}$ and $\rho=\rho_2=\text{const}$ for the case $n\equiv\text{const}$ gives zero, in agreement with the conclusion from Sec. IV.

Consider the case where there is a smooth layer without total internal reflection (the first one) which is separated from the TIR layer by a slab with $n=n_2=\text{const}$, $\rho=\rho_2=\text{const}$.

In that case the overall phase difference $\varphi(E_y)-\varphi(E_x)$ consists of a doubled contribution due to that first layer and due to the TIR layer. It is quite natural that the first contribution as calculated according to (55) equals just the doubled value of contribution of the same layer as calculated by integration of (25). That may be easily verified through integration by parts; doubling is connected with forth and back passage.

VIII. DISCUSSION

This paper is devoted to the calculation of small ($\sim\lambda/a$) correction to a phase of electromagnetic wave. In this connection we would like to discuss the very notion of that phase.

When a *plane* wave with linear polarization propagates through *homogeneous* medium, all the six complex values—E_x,E_y,E_z,H_x,H_y,H_z—have the same phase, and therefore it is not necessary to mention specially whose phase is discussed. In an inhomogeneous medium the situation is more complicated, and the phases of different Cartesian components of **E** and **H** may be slightly different: mutual shift may be about λ/a. This is one of the sources of the apparent dissimilarity of Eqs. (55) and (25) (including the $d\psi/dz$ term). However, if one starts and finishes at the homogeneous parts of the medium, then any ambiguity vanishes from the accumulated phase difference.

In conclusion, in this paper we obtained the explicit expressions for the phase difference $\varphi(E_y)-\varphi(E_x)$ acquired by two orthogonally polarized waves, which has the order of magnitude $\sim\lambda/a$ and appears due to effective birefringence of a locally isotropic layered medium with an arbitrary profile of $\varepsilon(z)$ and $\mu(z)$. It is important that the expression is valid (and convergent) even in the case of total internal reflection by a medium with smooth inhomogeneity. Our work to generalize these results to the case of three-dimensional smooth profiles of $\varepsilon(x,y,z)$ and $\mu(x,y,z)$ has not yet allowed us to make any definite conclusion.

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